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Credit VaR and Risk-Bucket Capital Rules: A Reconciliation

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I. Introduction

In its June 1999 consultative paper and subsequent public statements, the Basle Committee on Bank Supervision identified several key objectives for reform of bank regulatory capital. First and foremost, regulatory capital charges at the asset level must be aligned more closely with underlying risk. The failure of the 1988 Basle Accord to distinguish among assets of very different degrees of credit risk created the incentive to move low-risk assets off balance sheet. The financial innovations which arose in response to this incentive have undermined the effectiveness of regulatory capital rules (see, e.g., Jones, 2000) and thus led to current efforts towards reform. Second, as an international framework, the Accord should be able to accommodate a significant degree of diversity across banks in sophistication and business mix, and should be applicable across a variety of national supervisory and accounting regimes. Third, it must be feasible for supervisors to validate with reasonable confidence all significant inputs supplied by the bank. Finally, the Accord must be made more flexible in order to evolve with the state of best practice in risk management.

At present, it appears that a reformed Accord will be a risk-bucketing system of one form or another. In such a system, banking book assets are grouped into "buckets" which are presumed to be homogeneous. Associated with each bucket is a fixed capital charge per dollar of exposure. At a minimum, one would expect the bucketing system to partition assets by borrower rating, which would be externally given by rating agencies under some proposals and internally assigned under others; and by one or more proxies for seniority/collateral type, which determines loss severity in the event of default. More complex systems would partition assets by maturity, country/industry of borrower, and perhaps other characteristics. Regardless of the sophistication or fineness of the bucketing scheme, capital charges are *portfolio-invariant*, i.e., the capital charge on a given asset depends only on its own characteristics, and not the characteristics of the portfolio in which it is held. I take portfolio-invariance to be the essential property of risk-bucket capital rules.

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For regulatory purposes, risk-bucketing rules offer some significant advantages. The current Accord is itself a simple risk-bucketing framework. The reformed Accord could introduce additional bucketing criteria and make better use of information in borrower ratings, yet still be viewed as a natural extension of the current regime. Because the capital charge for a portfolio is simply a weighted sum of the dollars in each bucket, risk-bucketing systems are relatively simple to administer and do not impose complex reporting requirements. Validation problems are also limited in scope. Should the use of internal ratings be permitted, the most significant empirical challenge facing supervisors would likely concern the quality of default probability estimates for internal grades. Finally, risk-bucketing systems are widely used today in bank RAROC systems, even at many institutions which use more sophisticated approaches for portfolio management.

A much-discussed alternative to risk-bucketing would base regulatory capital on output from banks' internal value-at-risk ("VaR") models. Under the VaR paradigm, an institution holds capital in order to maintain a probability of survival over some fixed horizon (say, one year) at a targeted level (say, 99.9%). To be consistent with its target (denoted q), the institution must hold reserves and equity sufficient to cover up to the q^{th} quantile of the distribution of portfolio loss over the horizon. The purpose of the model is to estimate the portfolio loss distribution from a parsimonious set of asset characteristics.

In order to obtain a portfolio loss distribution, a model must generate a joint distribution over credit losses at the asset level. The latest generation of widely-used models gives structure to this problem by assuming that correlations across borrowers in credit events arise due to common dependence on a set of *systematic risk factors*. Implicitly or explicitly, these factors represent the sectoral shifts and macroeconomic forces that impinge to a greater or lesser extent on all firms in an economy. A natural property of these models is that the marginal capital required for a loan depends on how it affects diversification, and thus depends on what else is in the portfolio.

The Basle Committee undertook a detailed study (1999a) of how internal models might be used for setting regulatory capital. The Committee acknowledged that a carefully specified and calibrated VaR model can deliver a more accurate measure of portfolio credit risk than any risk-bucketing system, but found that the present state of model development could not ensure an acceptable degree of comparability across institutions and that data constraints prevent validation of key model parameters and assumptions.[1] It seems unlikely, therefore, that regulators will be prepared in the near- to medium-term to accept the use of internal models for setting regulatory capital. Nonetheless, regulators and industry practitioners appear in broad agreement that the new Accord should permit evolution towards an internal models approach as models and data improve.

This paper seeks to facilitate the evolution from risk-bucketing to internal models by asking whether and how it might be possible to derive risk-bucket capital rules within a well-specified risk-factor model. From a technical point of view, this is equivalent to asking: Under what assumptions does a risk-factor model yield portfolio-invariant marginal contributions to

1. In an industry practitioner response to the Basle consultative paper, GARP (1999) acknowledges the obstacles to immediate adoption of an internal models regulatory regime, but argues that the challenges can be met through an evolutionary, piecemeal approach to regulatory certification of model components.

value-at-risk? In Section II, I show within one widely-used VaR model that there are two necessary and sufficient conditions:

1. the portfolio must be asymptotically fine-grained, in the sense that no exposure in the portfolio can account for more than an arbitrarily small share of total portfolio exposure; and
2. there must be only a single systematic risk factor.

Extensions and implications of this results are discussed in Section III.

II. Asymptotic properties of a simple model

The essential intuitions behind the results of this paper are most easily conveyed in the context of a simple actuarial model of credit risk. Under an actuarial, or book value, definition of loss, credit loss arises only in the event of borrower default. Change in market value due to rating downgrade or upgrade is ignored. An especially tractable actuarial model is Credit Suisse Financial Products' *CreditRisk+*. In this section, I derive my results within a somewhat generalized version of this widely-used model.

A fundamental concept in *CreditRisk+*, as in any risk factor model, is the distinction between unconditional and conditional event probabilities. A borrower's *unconditional* probability of default ("PD"), also known as its expected default frequency, is the probability of default before a specified horizon given all information currently observable. The *conditional* default probability is the PD we would assign the borrower if we also knew what the realized value of the systematic risk factors at the horizon would be. The unconditional PD is the average value of the conditional default probability across all possible realizations of the systematic risk factors.

To gain some intuition for this terminology, consider a simple credit cycle in which the systematic risk factor takes only three values. The "bad state" corresponds to a recession at the risk horizon, the "good state" to an expansion, and the "neutral state" to ordinary times. Say that we currently are in a neutral state, and assign probabilities of ¼, ½ and ¼ to the three states (respectively) at the risk horizon. Consider a borrower who defaults with probability 2% in the event of a bad state, probability 1% in the neutral state, and probability 0.4% in the event of a good state. The "conditional default probability" is then 0.4%, 1%, or 2%, depending on which state we condition upon. The PD is the probability-weighted average of these values, or 1.1%.

Let X denote the systematic risk factors and x denote a realization of X. In the general case, X is multivariate, but for the time being I assume it is univariate. It is assumed that X is positive-valued with mean one and variance σ^2.[2] Conditional on X, the portfolio's remaining credit risk is assumed to be idiosyncratic to the individual borrowers. Let $p_i(x)$ denote the probability of default for borrower i conditional on $X{=}x$. In *CreditRisk+*, this is specified as

(1) $$p_i(x) = \bar{p}_i\left(1 + w_i(x-1)\right)$$

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2. In *CreditRisk+* it is assumed that X is gamma-distributed. For the purposes of this paper, I need only assume it has known distribution on $\Re^+$.

where $\bar{p}_i$ denotes the PD for borrower *i*. Note that $E[p_i(X)]=\bar{p}_i$ as required. Across the portfolio, the systematic factor pushes default probabilities above (below) their expected values when its realization is higher (lower) than its expected value, and thereby generates correlations in defaults. The sensitivity of borrower *i* to *X* is determined by the size of the factor loading $w_i \in [0,1]$. See CSFP (1997) and Gordy (2000a) on the *CreditRisk+* model and on its relationship to The RiskMetrics Groups' *CreditMetrics* model.

Imagine that the bank selects its portfolio as a large but finite subset of an infinite sequence of "potential" term loans, each to a unique borrower.[3] Let A_i denote the book value, or *exposure*, on the loan to borrower *i*.[4] To guarantee that idiosyncratic risk vanishes as more assets are added to the portfolio, it is necessary to place bounds on the A_i. For simplicity of exposition, I impose:

Assumption 1: Exposure sizes are bounded in some interval $[A^{min}, A^{max}]$, where $A^{min}>0$ and A^{max} is finite.

A much less restrictive assumption could be substituted; see Gordy (2000b) for an alternative. It is necessary only to prevent the A_i from blowing up too quickly toward infinity or shrinking too quickly towards zero.

Let D_i be a default indicator; i.e., D_i is a random variable with value 1 if borrower *i* defaults and value 0 otherwise. Loss given default ("LGD") by borrower *i* is a fraction λ_i of book value. Although *CreditRisk+* assumes that λ_i is fixed and known ex-ante, I allow for risk in LGD which is idiosyncratic to the borrower. Let $\bar{\lambda}_i$ denote the expected value of λ_i.

For a portfolio of the first *n* borrowers, define the *portfolio loss rate* L_n as the ratio of total losses to total portfolio exposure, i.e.,

(2)
$$L_n \equiv \frac{\sum_{i=1}^{n} \lambda_i A_i D_i}{\sum_{i=1}^{n} A_i}$$

Our first asymptotic result is:

Proposition 1: Conditional on $X=x$, $L_n - E[L_n|x] \rightarrow 0$, almost surely.

The result is essentially a corollary of a strong law of large numbers. Proof is given in the Appendix. Note that there is no restriction on the relationship between the exposure size for a facility and its quality (as measured by its PD or LGD parameters), so, for example, there is no problem if larger loans tend to be of higher or lower quality than smaller loans.

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3. This is without loss of generality. Under actuarial treatment of loss, multiple loans to a single borrower may be aggregated into a single loan without affecting the results.

4. If we allowed for unused lines of credit in the portfolio, then A_i would be the amount which would be drawn in the event of default. In practice, this might be difficult to estimate ex-ante.

In intuitive terms, Proposition 1 says that as the exposure share of each asset in the portfolio goes to zero, idiosyncratic risk in portfolio loss is diversified away perfectly. In the limit, the loss rate converges to a fixed function of the systematic factor X. I refer to this limiting portfolio as "asymptotically fine-grained." An implication is that, in the limit, we need only know the unconditional distribution of $E[L_n|X]$ to answer questions about the unconditional distribution of L_n. In particular, I next demonstrate that this applies to tail quantiles of the distribution of L_n, i.e., to value-at-risk.

One convenient property of the *CreditRisk+* framework is the linear form for the conditional expected value of the portfolio loss rate, which can be written as

$$\begin{aligned} E[L_n|x] &= \frac{\sum_{i=1}^n \bar{\lambda}_i A_i p_i(x)}{\sum_{i=1}^n A_i} = \frac{\sum_{i=1}^n \bar{\lambda}_i A_i \bar{p}_i(1+w_i(x-1))}{\sum_{i=1}^n A_i} \\ &= \frac{\sum_{i=1}^n \bar{\lambda}_i A_i \bar{p}_i}{\sum_{i=1}^n A_i} + (x-1)\frac{\sum_{i=1}^n \bar{\lambda}_i A_i \bar{p}_i w_i}{\sum_{i=1}^n A_i} \\ &= E[L_n] + \delta_n(x-1) \end{aligned} \tag{3}$$

where

$$\delta_n \equiv \frac{\sum_{i=1}^n \bar{\lambda}_i A_i \bar{p}_i w_i}{\sum_{i=1}^n A_i}. \tag{4}$$

The linear form allows us to decompose expected loss under any realization of the systematic factor X into two components, the unconditional expected loss (which is assumed to be covered by the loan loss reserve) and a systematic-risk loss to be covered by capital.

For the problem to be interesting, we need to bound $\delta_n>0$ as $n\rightarrow\infty$. If the expected LGDs, PDs or factor loadings were to converge to zero, then in the limit the conditional expectation $E[L_n|x]$ would converge to the unconditional expectation $E[L_n]$ for all x. No capital would be required beyond loan loss reserves. A minimally restrictive way to bound δ_n asymptotically is to impose:

Assumption 2: There exists a constant $\psi>0$ and finite integer M such that $\forall n>0$ there is at least one i in the set $\{n,n+1,...,n+M\}$ for which $\bar{\lambda}_i\bar{p}_i w_i \geq \psi$.

It is straightforward to show that Assumptions 1 and 2 together guarantee that for large enough n, $\delta_n > \underline{\delta} \equiv \psi A^{min}/(A^{min}+MA^{max})>0$. This leads to the main result of this paper. Letting x_q denote the q^{th} quantile of the distribution of X, we have

Proposition 2: As $n\rightarrow\infty$, $\Pr(L_n \leq E[L_n|x_q]) \rightarrow q$.

Proof is given in the Appendix. Under the single risk-factor assumption, this proposition implies a simple rule for setting required capital for an asymptotically fine-grained portfolio. To reduce the probability of insolvency to 1-q, the bank must hold $E[L_n|x_q]$ in reserves and capital. Assuming expected loss is covered by the loan loss reserve, then, by equation (3), capital must equal $\delta_n(x_q-1)$ per dollar of total exposure. Because δ_n can be expressed as a sum of asset-level terms, the capital requirement can be implemented at the asset level by setting the capital charge for loan i to $\bar{\lambda}_i\bar{p}_i w_i(x_q-1)$ per dollar of exposure to i. This "asymptotic capital charge" depends only on the assumed distribution of the systematic risk factor and on characteristics of loan i. Beyond the requirement that the portfolio be asymptotically fine-grained (and the technical restrictions of Assumptions 1 and 2), it imposes no restriction on the make-up of the remainder of the portfolio, and is therefore portfolio-invariant.

Proposition 2 demonstrates the sufficiency of the single risk factor assumption to deliver portfolio-invariant capital charges in an asymptotic portfolio. The necessity of both the single factor assumption and asymptotic granularity can also be demonstrated. Asymptotic granularity is required because, for any fixed n, idiosyncratic risk is not fully diversified. Consider the case of a homogeneous portfolio. The $\{\delta_n\}$ are constant across n, so the variance of the portfolio loss rate can be written as

$$V[L_n] = V[E[L_n|X]] + E[V[L_n|X]] = \delta^2\sigma^2 + (1/n)E[V[\lambda_1 D_1|X]]. \tag{5}$$

The marginal contribution of the n^{th} loan to the variance of the loss rate depends on n, and thus its marginal contribution to the quantiles of the loss rate distribution must in general depend on n as well. In the case of a finite heterogeneous portfolio, the marginal contribution of an asset to portfolio VaR depends in a complex manner on the size and composition of the rest of the portfolio. Therefore, no finite portfolio can support portfolio-invariant capital charges.

The single risk factor assumption also is unavoidable. Say that there were two independent systematic risk factors, X_1 and X_2, each gamma-distributed with mean one and variance σ^2. To keep the example simple, assume the portfolio has only two types of loans. Each type has the same exposure size, PD and expected LGD. Type 1 has factor loading w on X_1 and zero on X_2, and Type 2 has factor loading zero on X_1 and w on X_2. Assume that, as the portfolio grows, the ratio of m Type 2 loans to one Type 1 loan remains fixed. The proof of Proposition 1 does not require that X be univariate, so its holds for this two-factor world as well. Therefore, as $n\rightarrow\infty$, L_n converges in distribution to $E[L_\infty|X]$, which can be written as

$$E[L_\infty|X] = \bar{\lambda}\bar{p}\left(1 + w\left(\frac{X_1 + mX_2}{m+1} - 1\right)\right). \tag{6}$$

When m=0, there is effectively only one risk factor, so the appropriate asymptotic capital charge per dollar exposure for Type 1 is the same as in the single factor case. When m=1, it can be shown that $(X_1+mX_2)/(m+1)$ is itself gamma-distributed with mean one and variance $\sigma^2/2$. Quantiles of the loss rate distribution therefore can be calculated as if there were a single systematic factor with variance $\sigma^2/2$. The asymptotic capital charge per dollar exposure is the same for the two loan types (because of the symmetry when m=1), and is lower than the capital charge to Type 1 in the m=0 case. As m increases, the systematic risk in the portfolio is increasingly dominated by factor X_2 relative to X_1, so the asymptotic capital charge increases for Type 2 loans and decreases for Type 1 loans. The intuition readily generalizes to heterogeneous portfolios in more complex settings. Whenever there are multiple risk factors, the contribution of a given loan to portfolio VaR depends on whether the systematic risk associated with the borrower diversifies or further concentrates the systematic risk associated with other borrowers in the portfolio. Therefore, the appropriate asymptotic capital charge for the loan must depend on what else is in the portfolio.

III. Implications and Extensions

The results of this paper can be generalized significantly. In place of *CreditRisk+*, Gordy (2000b) shows that one can use any risk-factor model of portfolio credit risk, including multi-state mark-to-market models. We can allow for systematic risk in loss given default as well. Apart from minor technical restrictions, it is always the case that risk-factor VaR models yield portfolio-invariant capital charges if and only if (a) there is only a single systematic risk factor,[5] and (b) bank portfolios are asymptotically fine-grained (i.e., no exposure accounts for more than an arbitrarily small portion of total exposure).

The methods and conclusions presented here should not be unfamiliar to industry practitioners. Large-sample approximations have previously been applied to homogeneous portfolios under actuarial single factor versions of The RiskMetrics Group's *CreditMetrics* and KMV *Portfolio Manager* in order to obtain computational shortcuts (see Finger, 1999, and Vasicek, 1997, respectively). At a presentation in September 1999, Tom Wilde of Credit Suisse Financial Products used similar ideas to motivate a proposed "systematic risk contributions" approach to regulatory capital.

There are several implications for current efforts to reform the Basle Accord. First, it is indeed possible to make a risk-bucketing approach consistent with restricted versions of any of today's leading models of portfolio value-at-risk. Doing so would likely benefit the long-term goal of evolving towards models-based calculation of regulatory capital. For the near-term, the asymptotic single-factor approach offers an internally consistent framework within which to calibrate risk-bucket capital charges. In contrast to judgement-based approaches, a well-specified model gives regulators and banks a starting point for objective discussion of the sources of portfolio credit risk and the empirical basis for the levels of capital charges.

Second, this analysis suggests that "risk-bucketing" is something of a misnomer. Grouping assets into relatively homogeneous buckets might be convenient from an administrative point of view, but is not necessary for calibration of capital charges. Even if we assign default probabilities, factor loadings and expected recovery rates on a continuum of values, capital charges are portfolio-invariant in the asymptotic limit. In particular, it should be noted that there is no need for the number of assets to be very large in *each* bucket, but only in the portfolio as a whole.

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5. An important technical restriction is that conditional expected credit loss for each possible bank asset is monotonic in the realization of the systematic factor. Put another way, there can be no hedging assets in the banking book.

Finally, even though no bank can have an infinite number of loans, the asymptotic nature of the results does not diminish their practical usefulness. As portfolios grow larger, the discrepancy between an asset's appropriate marginal capital charge and its asymptotic capital charge diminishes to zero. Gordy (2000b) shows that VaR is well-approximated by its asymptotic value for reasonably-sized homogeneous portfolios. From an empirical point of view, the need to assume a single systematic risk factor is a more serious concern, because, in effect, it imposes a monolithic global business cycle on all borrowers. By assumption, all other credit risk is strictly idiosyncratic to the borrower. In reality, the global business cycle is a composite of many small economic changes, which might be tied to geography (e.g., political shifts, natural disasters) or to prices of production inputs (e.g., oil, metals). A single factor model cannot capture any clustering of firm defaults due to common sensitivity to these smaller-scale components of the global business cycle. Holding fixed the state of the global economy, a local recession in, for example, Spain is permitted to contribute nothing to the default rate of Spanish borrowers. If there are indeed pockets of risk, then calibrating a single risk factor model to a broadly diversified bank may significantly understate the capital needed to support a regional or specialized lender.

Appendix

Proof of Proposition 1 requires a version of the strong law of large numbers for a sequence $\{Y_n\}$ of random variables and a sequence $\{a_n\}$ of positive constants. Let V[Y] denote the variance of Y.

Lemma 1: If $a_n \uparrow \infty$ and $\sum_{n=1}^{\infty} (\mathrm{V}[Y_n]/a_n^2) < \infty$, then

$$\left(\sum_{i=1}^{n} Y_i - \mathrm{E}[\sum_{i=1}^{n} Y_i]\right)/a_n \to 0, \text{ a.s..} \tag{7}$$

Proof is given by Petrov (1995), Theorem 6.7.

To apply to Proposition 1, let $Y_n = \lambda_n A_n D_n$ and let $a_n = \sum_{i=1}^{n} A_i$. In economic terms, Y_n is the dollar amount lost on borrower n and a_n is the total exposure to the first n borrowers. Assumption 1 guarantees that $a_n \uparrow \infty$. To show that $\sum_{n=1}^{\infty} (\mathrm{V}[Y_n]/a_n^2) < \infty$, note that $\lambda_n D_n \in [0,1]$ implies $\mathrm{V}[\lambda_n D_n] < 1$, so

$$\begin{aligned} \sum_{n=1}^{\infty} \mathrm{V}[Y_n]/a_n^2 &= \sum_{n=1}^{\infty} \frac{\mathrm{V}[\lambda_n D_n] A_n^2}{(\sum_{i=1}^{n} A_i)^2} < \sum_{n=1}^{\infty} \left(\frac{\mathrm{A_n}}{\sum_{i=1}^{n} A_i} \right)^2 \\ &\le \sum_{n=1}^{\infty} \left(\frac{A^{\max}}{(n-1)A^{\min} + A^{\max}} \right)^2 < \infty \end{aligned} \tag{8}$$

where the second inequality follows from Assumption 2, and the final inequality can be checked using the integral test for series convergence. The conditions of Lemma 1 are therefore satisfied. The loss ratio L_n is equal to $\sum_{i=1}^{n} Y_i / a_n$ so Proposition 1 is proved.

I next prove Proposition 2.[6] For n sufficiently large, δ_n is bounded above zero, which guarantees that

$$\Pr(\mathrm{E}[L_n|X] \leq \mathrm{E}[L_n|x_q]) = \Pr(\delta_n(X - x_q) \leq 0) = \Pr(X \leq x_q) = q. \tag{9}$$

Equation (9) implies that $\Pr(L_n \leq \mathrm{E}[L_n|x_q]) \rightarrow q$ if and only if $\Pr(L_n \leq \mathrm{E}[L_n|x_q])$-$\Pr(\mathrm{E}[L_n|\mathrm{X}] \leq \mathrm{E}[L_n|x_q]) \rightarrow 0$. This difference in probabilities can be re-written as

$$\begin{aligned} &\Pr(L_n \leq \mathrm{E}[L_n|x_q]) - \Pr(\mathrm{E}[L_n|X] \leq \mathrm{E}[L_n|x_q]) \\ &\qquad = \mathrm{E}\left[1_{\{L_n \leq \mathrm{E}[L_n|x_q]\}}\right] - \mathrm{E}\left[1_{\{\mathrm{E}[L_n|X] \leq \mathrm{E}[L_n|x_q]\}}\right] \\ &\qquad = \mathrm{E}\left[\mathrm{E}\left[1_{\{L_n \leq \mathrm{E}[L_n|x_q]\}} - 1_{\{\mathrm{E}[L_n|X] \leq \mathrm{E}[L_n|x_q]\}} | X\right]\right] \equiv E[Z_n], \end{aligned} \tag{10}$$

where $\mathbf{1}_{\{statement\}}$ denotes the indicator function which equals 1 if *statement* is true and 0 otherwise. The difference in indicator functions in Z_n is nonzero only under two circumstances, (a) if $L_n \leq \mathrm{E}[L_n|x_q]$ and $\mathrm{E}[L_n|\mathrm{X}] > \mathrm{E}[L_n|x_q]$, or (b) if $L_n > \mathrm{E}[L_n|x_q]$ and $\mathrm{E}[L_n|\mathrm{X}] \leq \mathrm{E}[L_n|x_q]$. Treating case (a) first, observe that

$$\begin{aligned} &\mathrm{E}\left|1_{\{L_n \leq \mathrm{E}[L_n|x_q] \,\wedge\, \mathrm{E}[L_n|X] > \mathrm{E}[L_n|x_q]\}} | X\right| \\ &\qquad = \Pr(L_n \leq \mathrm{E}[L_n|x_q] | X) \cdot 1_{\{X > x_q\}} \\ &\qquad = \Pr(L_n - E[L_n|X] \leq \mathrm{E}[L_n|x_q] - E[L_n|X] | X) \cdot 1_{\{X > x_q\}} \\ &\qquad = \Pr(L_n - E[L_n|X] \leq \delta_n(x_q - X) | X) \cdot 1_{\{X > x_q\}} \\ &\qquad \leq \Pr(L_n - E[L_n|X] \leq \underline{\delta}(x_q - X) | X) \cdot 1_{\{X > x_q\}} \\ &\qquad \leq \Pr(|L_n - E[L_n|X]| \geq \underline{\delta}(X - x_q) | X) \cdot 1_{\{X > x_q\}}. \end{aligned} \tag{11}$$

Almost sure convergence implies convergence in probability (see White, 1984, Theorem 2.24), so we have by Proposition 1 that

$$\Pr(|L_n - E[L_n|X]| \geq \underline{\delta}(X - x_q) | X) \rightarrow 0 \tag{12}$$

for all $X > x_q$ as $n \rightarrow \infty$, which implies that

$$\mathrm{E}\left|1_{\{L_n \leq \mathrm{E}[L_n|x_q] \,\wedge\, \mathrm{E}[L_n|X] > \mathrm{E}[L_n|x_q]\}} | X\right| \rightarrow 0. \tag{13}$$

6. I am grateful to Darrell Duffie for suggesting this line of argument.

Using exactly the same argument, we obtain

$$\mathrm{E}\left[1_{\{L_n > \mathrm{E}[L_n|x_q] \,\wedge\, \mathrm{E}[L_n|X] \leq \mathrm{E}[L_n|x_q]\}} \middle| X\right] \to 0 \tag{14}$$

as well. Together, equations (13) and (14) imply that $Z_n \to 0$ as $n\to\infty$. By the dominated convergence theorem (see Billingsley, 1995, Theorem 16.4), the expression in equation (10) also must converge to zero, which completes the proof.

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